

# MD-STACKS

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**ABSTRACT.** We propose a generalisation of Mori dream spaces to stacks. We show that this notion is preserved under root constructions and taking abelian gerbes. Unlike the case of Mori dream spaces, such a stack is not always given as a quotient of the spectrum of its Cox ring by the Picard group. We give a criterion when this is true in terms of Mori dream spaces and root constructions. Finally, we compare this notion with the one of smooth toric Deligne-Mumford stacks.

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## INTRODUCTION

In [18], Yi Hu and Seán Keel introduce the notion of a *Mori dream space*  $X$ , which is defined in such a way that the Minimal Model Program works very nicely. Moreover, they show that such a  $X$  can be written as a GIT quotient of the spectrum of its *Cox ring*  $\mathcal{R}(X)$  by  $\mathrm{Hom}(\mathrm{Cl}(X), \mathbb{C}^*)$ , where  $\mathrm{Cl}(X)$  is assumed to be free of finite rank. In [17], Jürgen Hausen gives a way to allow also the case where  $\mathrm{Cl}(X)$  is just finitely generated. We want to note here, that the unstable locus of  $\mathrm{Spec} \mathcal{R}(X)$  is cut out by the so-called *irrelevant ideal*  $J_{\mathrm{irr}}$  which is generated by the global sections of an ample divisor on  $X$ .

In this paper, we propose a definition of Mori dream stacks as a generalisation of Mori dream spaces. For us, an *MD-stack*  $\mathcal{X}$  is a smooth Deligne-Mumford stack, with only constant invertible functions, finitely generated Picard group  $\mathrm{Pic}(\mathcal{X})$  and finitely generated Cox ring  $\mathcal{R}(\mathcal{X})$ . It turns out that an MD-stack  $\mathcal{X}$  is not automatically of the form

$$\mathcal{X} = [\mathrm{Spec} \mathcal{R}(\mathcal{X}) \setminus V(J_{\mathrm{irr}}) / \mathrm{Hom}(\mathrm{Pic}(\mathcal{X}), \mathbf{k}^*)],$$

as in the case of varieties. If  $\mathcal{X}$  is such a quotient, we call it an *MD-quotient*.

We show that MD-stacks are preserved by classical constructions for stacks, like the root constructions with simple normal crossing divisors and by taking abelian gerbes; see Theorems 2.9 and 2.10. MD-quotients are also preserved by these root constructions but not by arbitrary abelian gerbes, for latter we have to restrict ourselves to the special case of gerbes: the roots of line bundles; see Theorem 2.12.

The main result of this paper is the following characterisation of MD-stacks and of the subclass of MD-quotients. Let  $\mathcal{X}$  be a Deligne-Mumford stack and  $\mathcal{X} \rightarrow X$  be the structure morphism to its *coarse moduli space*  $X$ . This morphism factors through  $\mathcal{X} \rightarrow \mathcal{X}^{can} \rightarrow X$ , where  $\mathcal{X}^{can}$  is the *canonical stack* i.e. the minimal orbifold having  $X$  as coarse moduli space.

**Main Theorem** (Theorems 3.1 and 3.5). *Let  $\mathcal{X}$  be a smooth Deligne-Mumford stack and  $\mathcal{X} \rightarrow X$  be the structure morphism to its coarse moduli space  $X$ . Then:*

- (1)  *$\mathcal{X}$  is an MD-stack if and only if  $X$  is an MD-space.*
- (2)  *$\mathcal{X}$  is an MD-quotient if and only if  $X$  is an MD-space with the property that  $\mathrm{Spec} \mathcal{R}(X) \setminus V(J_{\mathrm{irr}})$  is smooth and  $\mathcal{X}$  can be obtained from  $\mathcal{X}^{can}$  by root constructions with simple normal crossing divisors and line bundles.*

The easiest class of Mori dream spaces are toric varieties. They are characterised among the Mori dream spaces by the property that their Cox rings are polynomial rings. Moreover a Mori dream space can be embedded as a closed subvariety in a toric variety in such a way that their Picard groups are naturally isomorphic; see [18]. In [8], Lev Borisov, Linda Chen and Greg Smith introduce toric stacks. Under the subsequent work, in [12] Barbara Fantechi, Étienne Mann and Fabio Nironi give a geometric definition of a smooth toric Deligne-Mumford stack with an action of a Deligne-Mumford torus. They show that any such stack can be obtained from its coarse moduli space, which is again toric, by root constructions along torus-invariant divisors and line bundles. Many of our results are a generalisation of their ideas to the MD-setting.

The last part of our article is dedicated to the relation between MD-quotient and smooth toric Deligne-Mumford stacks. As a generalisation of [18] mentioned above, we show that an MD-quotient is a smooth toric Deligne-Mumford stack if and only if its Cox ring is a polynomial ring; see Theorem 4.1. Moreover, under the hypothesis that the divisor class group of its coarse moduli space is free, Theorem 4.2 states that an MD-quotient can be embedded in a smooth toric Deligne-Mumford stack.

**Notations and Conventions.** In this article,  $\mathbf{k}$  will always be an algebraically closed field of characteristic 0. A stack is always meant to be algebraic, separated and of finite type over  $\mathbf{k}$ . We work in the étale topology.

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## 1. PRELIMINARIES

**1.1. MD-spaces and Cox rings.** Our definition of a Mori dream space will differ slightly from the standard terminology coming from the Minimal Model Program. To avoid confusion, we will always use the abbreviation MD-space when referring to the definition given below. The reason is that we rely on a quotient description of such a variety, which comes in handy when passing to stacks. Therefore, our definition will be between the one for Mori dream spaces found in [18] and the characterisation of varieties which are GIT quotients of the spectrum of its (finitely generated) Cox ring as found in [5].

**Definition 1.1.** A normal variety  $X$  will be called a *MD-space* if

- (1)  $X$  has at most quotient singularities,
- (2)  $H^0(X, \mathcal{O}_X^*) = \mathbf{k}^*$ ,
- (3)  $\mathrm{Cl}(X)$  is finitely generated as a  $\mathbb{Z}$ -module,
- (4) its Cox ring  $\mathcal{R}(X)$  is finitely generated as a  $\mathbf{k}$ -algebra.

**Remark 1.2.** Note that a variety has by definition *quotient singularities* if there exists a analytically open cover by sets of the form  $U/G$ , where  $U$  is smooth and  $G$  a finite group. Consequently such a variety is  $\mathbb{Q}$ -factorial.

If  $X$  in the definition is moreover projective, then the definition is a bit more restrictive than the standard one, where worse singularities are allowed as long as the variety is still  $\mathbb{Q}$ -factorial; see [18, Proposition 2.9].

Let  $X$  be an MD-space. For us the most important feature is that  $X$  has a description as a GIT quotient of the spectrum of its Cox ring  $\mathcal{R}(X)$  by  $\mathrm{Hom}(\mathrm{Cl}(X), \mathbf{k}^*)$ . Let us introduce all the ingredients to understand this quotient description. The Cox ring can be naïvely defined as the  $\mathrm{Cl}(X)$ -graded vector space

$$\mathcal{R}(X) = \bigoplus_{L \in \mathrm{Cl}(X)} H^0(X, L),$$

but as pointed out in [18], this definition does not admit a well-defined ring structure. To remedy this in the case when  $\mathrm{Cl}(X)$  is free, one has to fix a basis of  $\mathrm{Cl}(X)$ . Then the product can be defined by multiplying the sections as rational functions.

For an arbitrary finitely generated  $\mathrm{Cl}(X)$ , this approach also works although it is a bit more technical; see [17]. In the following presentation we will follow [5, Section I.4].

Choose a finite number of divisors  $D_1, \dots, D_m$ , such that the classes  $[D_1], \dots, [D_m]$  generate  $\mathrm{Cl}(X)$ . Denote by  $K$  the free subgroup of the group of Weil divisors  $\mathrm{WDiv}(X)$  generated by these divisors. So we have a surjection  $K \twoheadrightarrow \mathrm{Cl}(X)$ . If we write  $K^0$  for its kernel, then we obtain a short exact sequence  $K^0 \hookrightarrow K \twoheadrightarrow \mathrm{Cl}(X)$  with  $K, K^0$  free groups. Finally, since the composition of  $K^0 \hookrightarrow K \hookrightarrow \mathrm{WDiv}(X)$  and  $\mathrm{WDiv}(X) \twoheadrightarrow \mathrm{Cl}(X)$  is zero, the map  $K^0 \hookrightarrow \mathrm{WDiv}(X)$  has to factor uniquely over  $\mathbf{k}(X)^*$  that is the kernel of  $\mathrm{WDiv}(X) \twoheadrightarrow \mathrm{Cl}(X)$ . So we obtain a map  $\chi: K^0 \hookrightarrow \mathbf{k}(X)^*$ . Especially for a principal divisor  $E \in K^0$  we have  $E = \mathrm{div} \chi(E)$ .

Consider the sheaf of  $\mathrm{Cl}(X)$ -graded rings  $\mathcal{S} = \bigoplus_{D \in K} \mathcal{O}_X(D)$  containing the homogeneous ideal  $\mathcal{I} = \langle 1 - \chi(E) \mid E \in K^0 \rangle$ .

**Definition 1.3.** The *Cox ring* of  $X$  is the  $\mathrm{Cl}(X)$ -graded ring

$$\mathcal{R}(X) := H^0(X, \mathcal{S})/H^0(X, \mathcal{I}).$$

We want to note that the definition of the Cox ring depends on some choices. But for different choices, the resulting rings will be (non-canonically) isomorphic.

The  $\mathrm{Cl}(X)$ -grading of the Cox ring  $\mathcal{R}(X)$  is equivalent to an action of  $\mathrm{Hom}(\mathrm{Cl}(X), \mathbf{k}^*)$  on  $\mathrm{Spec} \mathcal{R}(X)$ . Actually,  $X$  becomes a GIT quotient by this action.

**Proposition 1.4** ([18, Proposition 2.9], [5, Section I.6]). *Let  $X$  be an MD-space. Then  $X$  is isomorphic to a GIT quotient  $\mathrm{Spec} \mathcal{R}(X) // \mathrm{Hom}(\mathrm{Cl}(X), \mathbf{k}^*)$ . More explicitly, for any ample line bundle  $L$  on  $X$ , define the irrelevant ideal  $J_{\mathrm{irr}} = J_{\mathrm{irr}}(X) := \sqrt{\langle H^0(X, L) \rangle}$ , that turns out to be independent of the choice of the ample line bundle. Then we obtain*

$$X = \mathrm{Spec} \mathcal{R}(X) \setminus V(J_{\mathrm{irr}}) / \mathrm{Hom}(\mathrm{Cl}(X), \mathbf{k}^*).$$

**1.2. Stacks.** We recall that we assume stacks to be algebraic, separated and of finite type over  $\mathbf{k}$ . Most of the constructions of this section will hold also if the stacks are not Deligne-Mumford. In this section we recall some basic facts on stacks, that will be useful for the following.

The first notion we need is that of a coarse moduli space. The main references for us are [22], [23] and [25].

**Definition 1.5.** Let  $\mathcal{X}$  be a stack. A *coarse moduli space* for  $\mathcal{X}$  is an algebraic space  $X$  over  $\mathbf{k}$  with a morphism  $\pi: \mathcal{X} \rightarrow X$ , such that:

- (1) universal property: for every morphism  $\psi: \mathcal{X} \rightarrow Z$  to an algebraic space, there exists a unique morphism  $f: X \rightarrow Z$ , such that  $\psi = f \circ \pi$ ;
- (2) the map  $|\mathcal{X}(\mathbf{k})| \rightarrow X(\mathbf{k})$  is bijective, where  $|\mathcal{X}(\mathbf{k})|$  denotes the set of isomorphism classes in  $\mathcal{X}(\mathbf{k})$ .

The following proposition actually characterises *good moduli spaces*, but in our situation the two notions coincide. Note that since we assume our stacks and varieties to be of finite type over  $\mathbf{k}$ , maps between them are automatically quasi-compact.

**Proposition 1.6** ([3], [4]). *Let  $\pi: \mathcal{X} \rightarrow X$  be the natural map to the coarse moduli space. Then*

- (3)  $\pi$  is proper;
- (4)  $\pi_*: \mathrm{Coh} \mathcal{X} \rightarrow \mathrm{QCoh} X$  is exact;
- (5) there is a natural isomorphism  $\mathcal{O}_X \rightarrow \pi_* \mathcal{O}_{\mathcal{X}}$ .

*Conversely, if  $\pi: \mathcal{X} \rightarrow X$  satisfies the conditions (2), (4) and (5), then  $\pi$  is a coarse moduli map.*

The following theorem assures the existence of coarse moduli spaces.

**Proposition 1.7** ([19], [11, Theorem 1.1]). *Let  $\mathcal{X}$  be a Deligne-Mumford stack. Then there exists a coarse moduli space  $\pi: \mathcal{X} \rightarrow X$  which is proper and quasi-finite. Moreover, if  $X' \rightarrow X$  is a flat map of algebraic spaces, then  $\mathcal{X} \times_X X' \rightarrow X'$  is a coarse moduli space too.*

Now we recall the concept of rigidification of a stack. We refer to [1], [3] and [12] for details.

Let  $\mathcal{X}$  be a stack. The *inertia stack* of  $\mathcal{X}$  is defined to be the fibre product  $I(\mathcal{X}) := \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$ . The inertia stack of a smooth Deligne-Mumford stack is smooth, but will consist of several components not all of the same dimension. Let  $I(\mathcal{X}) \rightarrow \mathcal{X}$  be the natural projection, its identity section gives an irreducible component canonically isomorphic to  $\mathcal{X}$ ; all other components are called *twisted sectors*.

A smooth Deligne-Mumford stack of dimension  $d$  is an *orbifold* if and only if all the twisted sectors have dimension  $d - 1$ , and is *canonical* if and only if all twisted sectors have dimension  $d - 2$ .

The canonical stack can be defined also as the smooth Deligne-Mumford stack  $\mathcal{X}^{can}$ , whose natural map to the coarse moduli space  $\pi: \mathcal{X}^{can} \rightarrow X$  fails to be an isomorphism just on a locus of dimension  $\leq d - 2$ . Note that  $\mathcal{X}^{can}$  is unique up to isomorphism with this property; see [12, Theorem 4.6].

The *generic stabiliser*  $I^{gen}(\mathcal{X})$  of  $\mathcal{X}$  is the subsheaf of groups of  $I(\mathcal{X})$ , given by the union of all  $d$ -dimensional components of  $I(\mathcal{X})$ .

Let now  $H \subset I(\mathcal{X})$  be a flat subgroup stack. Let  $\xi \in \mathcal{X}(T)$  be an object of the stack  $\mathcal{X}$  over a  $k$ -scheme  $T$ , which corresponds to a morphism  $T \rightarrow \mathcal{X}$ . The pullback  $T \times_{\mathcal{X}} I(\mathcal{X})$  is canonically isomorphic to the group scheme  $Aut_{T, \mathcal{X}}(\xi)$  and the pullback of  $H$  to  $T$  gives a flat subgroup scheme  $H_{\xi} \subset Aut_{T, \mathcal{X}}(\xi)$ . It is easy to see that the construction implies that  $H_{\xi}$  is normal in  $Aut_{T, \mathcal{X}}(\xi)$ . In this situation the following theorem defines the *rigidification* of  $\mathcal{X}$  by  $H$ .

**Proposition 1.8** ([3, Theorem A.1.]). *There is a stack  $\mathcal{X} // H$  called the rigidification of  $\mathcal{X}$  by  $H$  and a morphism  $\nu: \mathcal{X} \rightarrow \mathcal{X} // H$  satisfying the following properties:*

- (1) *for any  $k$ -scheme  $T$  and for any object  $\xi \in \mathcal{X}(T)$ , the homomorphism of group schemes  $Aut_{T, \mathcal{X}}(\xi) \rightarrow Aut_{T, \mathcal{X} // H}(\nu(\xi))$  is surjective with kernel  $H_{\xi}$ ,*
- (2)  *$\mathcal{X}$  is a fppf gerbe over  $\mathcal{X} // H$  (for the definition of gerbe, see Section 1.4). Furthermore, if  $\mathcal{X}$  is a Deligne-Mumford stack, then also  $\mathcal{X} // H$ .*

It can be shown that these properties characterise  $\mathcal{X} // H$  uniquely up to isomorphism.

A more intuitive way to think of the rigidification is to consider first the prestack  $(\mathcal{X} // H)^{pre}$ , whose objects are the same as those of  $\mathcal{X}$  and whose sheaf of automorphisms  $Aut_{T, (\mathcal{X} // H)^{pre}}(\xi)$  of an object  $\xi \in \mathcal{X}(T)$  is the quotient of  $Aut_{T, \mathcal{X}}(\xi)$  by the subsheaf of normal groups  $H_{\xi}$ . The rigidification  $\mathcal{X} // H$  can be obtained as the fppf stackification of this prestack. From this construction follows that  $\mathcal{X}$  and  $\mathcal{X} // H$  have the same coarse moduli space.

In our work we are mainly interested in the rigidification of a stack by its generic stabiliser, which we assume to be abelian and flat. It will be called the rigidification of  $\mathcal{X}$  and indicated by  $\mathcal{X}^{rig}$ .

**Proposition 1.9** ([12, Remark 3.7]). *Let  $\mathcal{X}$  be a Deligne-Mumford stack and  $\pi: \mathcal{X} \rightarrow X$  its coarse moduli map. Moreover, let  $D$  be a prime divisor on  $X$ . Denote by  $\mathcal{D}$  the reduced closed substack with support  $\pi^{-1}(D)$ . When restricting to the smooth locus of  $X$ ,  $D$  becomes Cartier, and there is a*

unique positive integer  $a$  such that  $\pi^{-1}(D \cap X_{\text{sm}}) = a(\mathcal{D} \cap \pi^{-1}(X_{\text{sm}}))$ . We call  $a$  the divisor multiplicity of  $\mathcal{D}$ .

**Definition 1.10.** Let  $\mathcal{X}$  be a Deligne-Mumford orbifold. The *ramification divisor* of  $\mathcal{X}$  is the sum of all prime divisors of  $\mathcal{X}$  with positive divisor multiplicity.

Now let  $\mathcal{X}$  be an arbitrary Deligne-Mumford stack and  $\nu: \mathcal{X} \rightarrow \mathcal{X}^{\text{rig}}$  its rigidification. If  $\mathcal{D}^{\text{rig}}$  is the ramification divisor of  $\mathcal{X}^{\text{rig}}$ , we call the reduced closed substack  $\mathcal{D}$  inside  $\nu^{-1}(\mathcal{D}^{\text{rig}})$  the ramification divisor of  $\mathcal{X}$ .

**Definition 1.11.** A line bundle  $\mathcal{L}$  on a stack  $\mathcal{X}$  is called (*cohomologically*) *ample* if  $H^i(\mathcal{X}, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$  vanishes for any sheaf  $\mathcal{F}$  on  $\mathcal{X}$ ,  $i > 0$  and  $n \gg 0$ .

**Lemma 1.12.** Let  $\pi: \mathcal{X} \rightarrow X$  be the map of a Deligne-Mumford stack to its coarse moduli space and  $L$  an ample line bundle on  $X$ . Then  $\pi^*L$  is an ample line bundle on  $\mathcal{X}$ .

*Proof.* Since  $L$  is locally free, we can use an underived projection formula to get  $\pi_*(F \otimes \pi^*L^{\otimes n}) = \pi_*F \otimes L^{\otimes n}$ . Applying on both sides  $H^i(X, -)$  this becomes  $H^i(\mathcal{X}, F \otimes \pi^*L^{\otimes n}) = H^i(X, \pi_*F \otimes L^{\otimes n})$ . But the latter cohomology group vanishes for  $i > 0$  and  $n \gg 0$ .  $\square$

**1.3. Root constructions.** In this section, we recall the definition of two root constructions as found in [10] and [2] and we state some facts about these.

**Definition 1.13.** Let  $\mathcal{X}$  be a stack and  $\mathcal{D}$  an effective divisor on  $\mathcal{X}$ . Let  $r$  be a positive integer. Consider the fibre product

$$\begin{array}{ccc} \mathcal{X} \times_{[\mathbb{A}^1/\mathbf{k}^*]} [\mathbb{A}^1/\mathbf{k}^*] & \longrightarrow & [\mathbb{A}^1/\mathbf{k}^*] \\ \downarrow & & \downarrow r \\ \mathcal{X} & \xrightarrow{\mathcal{D}} & [\mathbb{A}^1/\mathbf{k}^*] \end{array} \quad (1)$$

where the lower map corresponds to the divisor and the right map is induced by  $p \mapsto p^r$ . The stack  $\sqrt[r]{\mathcal{D}/\mathcal{X}} := \mathcal{X} \times_{[\mathbb{A}^1/\mathbf{k}^*]} [\mathbb{A}^1/\mathbf{k}^*]$  is the  $r$ -th root of the divisor  $\mathcal{D}$  on  $\mathcal{X}$ .

**Definition 1.14.** Let  $\mathcal{X}$  be a stack and  $\mathcal{L}$  a line bundle on  $\mathcal{X}$ . Let  $r$  be a positive integer. Consider the fibre product

$$\begin{array}{ccc} \mathcal{X} \times_{\mathcal{B}\mathbf{k}^*} \mathcal{B}\mathbf{k}^* & \longrightarrow & \mathcal{B}\mathbf{k}^* \\ \downarrow & & \downarrow r \\ \mathcal{X} & \xrightarrow{\mathcal{L}} & \mathcal{B}\mathbf{k}^* \end{array} \quad (2)$$

where the lower map corresponds to the line bundle and the right map is induced by  $p \mapsto p^r$ . The stack  $\sqrt[r]{\mathcal{L}/\mathcal{X}} := \mathcal{X} \times_{\mathcal{B}\mathbf{k}^*} \mathcal{B}\mathbf{k}^*$  is the  $r$ -th root of the line bundle  $\mathcal{L}$  on  $\mathcal{X}$ .

**Definition 1.15.** Let  $\mathcal{X}$  be a stack. We say that  $\mathcal{X}'$  is *obtained by roots* from  $\mathcal{X}$ , if  $\mathcal{X}'$  is the result of performing a finite sequence of root constructions with divisors or line bundles starting from  $\mathcal{X}$ .

According to [25, Appendix], a coherent sheaf on a Deligne-Mumford quotient stack  $\mathcal{X} = [Z/G]$  is a  $G$ -equivariant sheaf on  $Z$ , i.e. a coherent sheaf  $L_Z$  on  $Z$  equipped, for every  $g \in G$ , with an isomorphism  $\varphi_g: L_Z \rightarrow g^* L_Z$ , such that  $\varphi_{gh} = h^* \varphi_g \circ \varphi_h$ . The data  $\{\varphi_g\}_{g \in G}$  is called a  $G$ -linearisation of  $L_Z$ .

In the case of invertible sheaves this leads to the following exact sequence; see [20, Lemma 2.2]:

$$0 \rightarrow H_{alg}^1(G, \mathcal{O}_Z^*) \rightarrow \text{Pic}(\mathcal{X}) \rightarrow \text{Pic}(Z),$$

where the group  $H_{alg}^1(G, \mathcal{O}_Z^*)$  parametrises  $G$ -linearisations of the trivial line bundle on  $Z$ . The group of  $G$ -linearisations is naturally linked to the group of characters of  $G$  by the following exact sequence

$$1 \rightarrow H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^*) \rightarrow H^0(Z, \mathcal{O}_Z^*) \rightarrow \text{Hom}(G, \mathbf{k}^*) \rightarrow \text{Pic}(\mathcal{X}) \rightarrow \text{Pic}(Z),$$

from which we get that, if  $H^0(Z, \mathcal{O}_Z^*) = \mathbf{k}^*$ , the first arrow is an isomorphism and the group  $H_{alg}^1(G, \mathcal{O}_Z^*)$  coincides with the groups of characters of  $G$ .

In this case we can generalise [12, Lemma 7.1] to:

**Proposition 1.16.** *Let  $Z$  be a variety with  $H^0(Z, \mathcal{O}^*) = \mathbf{k}^*$  and  $G$  an abelian group acting on it, such that  $\mathcal{X} = [Z/G]$  is a Deligne-Mumford stack. Let  $\mathcal{D}$  be a divisor on  $\mathcal{X}$  corresponding to a line bundle  $\mathcal{L}$  and a section  $s$  of it. Assume that  $\mathcal{L}$  is given by the trivial line bundle on  $Z$  and a character  $\chi$  of  $G$ . Finally, let  $r$  be a positive integer.*

- (1) *The stack  $\sqrt[r]{\mathcal{D}/\mathcal{X}}$  is isomorphic to  $[Z'/G']$  where both  $Z'$  and  $G'$  are defined as the fibre products*

$$\begin{array}{ccc} Z' & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow r \\ Z & \xrightarrow{s} & \mathbb{A}^1 \end{array} \quad \begin{array}{ccc} G' & \longrightarrow & \mathbf{k}^* \\ \downarrow & & \downarrow r \\ G & \xrightarrow{\chi} & \mathbf{k}^* \end{array} \quad (3)$$

where both the maps on the right are induced by  $p \mapsto p^r$ . The action of  $G'$  on  $Z'$  is given by

$$(g, \lambda) \cdot (z, t) = (g \cdot z, \lambda t).$$

- (2) *The stack  $\sqrt[r]{\mathcal{L}/\mathcal{X}}$  is isomorphic to  $[Z/G']$  where  $G'$  is defined as above and acts on  $Z$  using the map  $G' \rightarrow G$ .*

**Remark 1.17.** The variety  $Z'$  together with the map  $Z' \rightarrow Z$  is called the  $r$ -th cyclic cover of  $Z$  ramified along  $D$ , which is the divisor  $\mathcal{D}$  pulled back to  $Z$ .

Note that, if  $\text{Pic}(Z) = 0$ , any invertible sheaf on the quotient stack  $\mathcal{X}$  is given by a unique character of  $G$ .

**Remark 1.18.** Let  $\mathcal{X}'$  be given as a  $r$ -th root of a divisor  $\mathcal{D}_i$  on  $\mathcal{X}_i$  with  $i = 1, 2$ . Let  $\nu_i: \mathcal{X}' \rightarrow \mathcal{X}_i$  be the corresponding map. Suppose  $\nu_i^* \mathcal{D}_i = r \mathcal{D}'$  for a divisor  $\mathcal{D}'$  on  $\mathcal{X}'$ . Then there is an isomorphism  $g: \mathcal{X}_1 \rightarrow \mathcal{X}_2$  with  $g \circ \nu_1 = \nu_2$ . In this sense, the base of a root construction is unique.

**Proposition 1.19** ([10, Theorem 2.3.3 and Remark]). *Let  $\mathcal{X}$  be a stack obtained by roots from a Deligne-Mumford stack. Then  $\mathcal{X}$  is also a Deligne-Mumford stack.*

**Proposition 1.20** ([4, Proposition 3.10(vii)]). *Consider the following cartesian square of Artin stacks.*

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{Y}' \\ f' \downarrow & & \downarrow f \\ \mathcal{X} & \longrightarrow & \mathcal{Y} \end{array}$$

*If  $f$  is cohomologically affine (i.e. quasi-compact and  $f_*$  is exact) and  $\mathcal{Y}$  has quasi-affine diagonal over  $\mathbf{k}$ , then  $f'$  is also cohomologically affine. Note that the condition on  $\mathcal{Y}$  is automatically fulfilled, if  $\mathcal{Y}$  is a Deligne-Mumford stack.*

**Lemma 1.21.** *Let  $\mathcal{X} \rightarrow X$  be the map to the coarse moduli space of a Deligne-Mumford stack  $\mathcal{X}$ , and  $\mathcal{X}'$  obtained by roots with divisors from  $\mathcal{X}$ . Then the composition  $\mathcal{X}' \rightarrow \mathcal{X} \rightarrow X$  is the coarse moduli map of  $\mathcal{X}'$ .*

*Proof.* Let  $\mathcal{X}'$  be obtained by one single root of a divisor from  $\mathcal{X}$ . So  $\mathcal{X}'$  fits into the following cartesian square:

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & [\mathbb{A}^1/\mathbf{k}^*] \\ \nu \downarrow & & \downarrow r \\ \mathcal{X} & \longrightarrow & [\mathbb{A}^1/\mathbf{k}^*] \end{array}$$

We will check that  $\mathcal{X}' \xrightarrow{\nu} \mathcal{X} \rightarrow X$  is a coarse moduli map, using the characterisation in Proposition 1.6. Since this already holds for  $\mathcal{X} \rightarrow X$ , we are left to show that  $\nu$  is cohomologically affine, induces an isomorphism  $\mathcal{O}_{\mathcal{X}} \rightarrow \nu_* \mathcal{O}_{\mathcal{X}'}$  and a bijection  $|\mathcal{X}'(\mathbf{k})| \rightarrow |\mathcal{X}(\mathbf{k})|$ .

First we show that  $\nu$  is cohomologically affine by applying Proposition 1.20. For this note that  $[\mathbb{A}^1/\mathbf{k}^*]$  has a quasi-affine diagonal over  $\mathbf{k}$ . Moreover, the composition  $[\mathbb{A}^1/\mathbf{k}^*] \xrightarrow{r} [\mathbb{A}^1/\mathbf{k}^*] \rightarrow \mathrm{Spec} \mathbf{k}$  is the same map as  $[\mathbb{A}^1/\mathbf{k}^*] \rightarrow \mathcal{B}\mathbf{k}^* \rightarrow \mathrm{Spec} \mathbf{k}$ , that is cohomologically affine. Therefore, by [4, Proposition 3.14], also  $[\mathbb{A}^1/\mathbf{k}^*] \xrightarrow{r} [\mathbb{A}^1/\mathbf{k}^*]$  is cohomologically affine, and consequently  $\nu$ , too.

Secondly,  $\mathcal{O}_{\mathcal{X}} \rightarrow \nu_* \mathcal{O}_{\mathcal{X}'}$  is an isomorphism, by [10, Theorem 3.1.1(3)].

Finally, we need to see whether  $\nu$  induces a bijection of closed points  $|\mathcal{X}'(\mathbf{k})| \rightarrow |\mathcal{X}(\mathbf{k})|$ . Let  $\mathrm{pt} \rightarrow \mathcal{X}$  be a closed point and consider the fibre product

$$\begin{array}{ccc} P & \longrightarrow & \mathcal{X}' \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & \mathcal{X} \end{array}$$

By extending this cartesian square by the diagram above, one sees that  $P$  is either  $\mathrm{pt}$  or  $[\mathrm{Spec}(\mathbf{k}^{[t]}/_{t^r})/\mu_r]$ , which has  $\mathrm{pt}$  or  $\mathrm{Spec}(\mathbf{k}^{[t]}/_{t^r})$  as an atlas. Note that latter contains a unique closed point. Therefore,  $|\mathcal{X}'(\mathbf{k})| \rightarrow |\mathcal{X}(\mathbf{k})|$  is surjective.

Suppose we have two maps  $f_i: \mathrm{pt} \rightarrow \mathcal{X}'$  with  $i = 1, 2$  such that the composition  $\mathrm{pt} \rightarrow \mathcal{X}' \rightarrow \mathcal{X}$  is the same map  $g$  for both up to isomorphism.



Therefore, we can build the following diagram where the square is cartesian

$$\begin{array}{ccc}
 \text{pt} & \xrightarrow{f_i} & \mathcal{X}' \\
 \downarrow \text{id} & \searrow & \downarrow \\
 P & \xrightarrow{\quad} & \mathcal{X}' \\
 \downarrow & & \downarrow \\
 \text{pt} & \xrightarrow{g} & \mathcal{X}
 \end{array}$$

So the  $f_i$  factor over  $P$ . Independent of its actual form,  $|P(\mathbf{k})| = \text{pt}$ , so the maps  $f_i$  differ only by an isomorphism. This shows that  $|\mathcal{X}'(\mathbf{k})| \rightarrow |\mathcal{X}(\mathbf{k})|$  is also injective.  $\square$

**Remark 1.22.** A very similar proof also works for roots with line bundles. But we will show the analogous statement in a more general situation, namely for gerbes, see Example 1.29.

**Lemma 1.23** ([24, Proposition 3.1]). *Let  $Z$  be a smooth variety and  $D_1, \dots, D_l$  prime divisors on  $Z$ . Moreover, let  $Z'$  be obtained from  $Z$  by cyclic covers along the divisors  $D_1, \dots, D_l$ . Then  $Z'$  is smooth if and only if  $D_1, \dots, D_l$  are simple normal crossing.*

**Corollary 1.24.** *Let  $\mathcal{X}$  be a smooth Deligne-Mumford stack and  $\mathcal{X}'$  obtained by roots along the divisors  $D_1, \dots, D_l$ . Then  $\mathcal{X}'$  is smooth if and only if  $D_1, \dots, D_l$  are normal crossing.*

*Proof.* Note that the smooth Deligne-Mumford stack  $\mathcal{X}$  can be étale locally covered by smooth varieties. Moreover, by Proposition 1.16, the root on such a variety  $U$  along some of its divisors yield a quotient stack, whose atlas is a cyclic cover of  $U$ . The application of Lemma 1.23 concludes the proof.  $\square$

**1.4. Gerbes.** In this section we recall the definition of gerbes and some of their properties. We mainly follow [9] and we refer to [14, Chapter IV.2] for a complete treatment.

**Definition 1.25.** A *gerbe* on a stack  $\mathcal{X}$  is a stack in groupoids  $\mathcal{G}$  on  $\mathcal{X}$  which is:

- (1) *locally non-empty*: for any atlas  $U \rightarrow \mathcal{X}$  of  $\mathcal{X}$ ,  $\mathcal{G}(U)$  is non-empty;
- (2) *locally transitive*: for any  $x, y \in \mathcal{G}(U)$  there is an open cover  $\{V_i \rightarrow U\}$ , such that  $x|_{V_i}, y|_{V_i} \in \mathcal{G}(V_i)$  are isomorphic.

Let  $G$  be a sheaf of abelian groups over  $\mathcal{X}$ . A gerbe  $\mathcal{G}$  on  $\mathcal{X}$  is called  *$G$ -banded* (or simply a  *$G$ -gerbe*) if, for every étale atlas  $U \rightarrow \mathcal{X}$  and for every object  $x \in \mathcal{G}(U)$ , there exists an isomorphism  $\alpha_x: G|_U \rightarrow \text{Aut}_{U, \mathcal{G}}(x)$  of sheaves of groups and all these isomorphisms are compatible with the pullbacks in the following sense. Given two objects  $x \in \mathcal{G}(U)$  and  $y \in \mathcal{G}(V)$  and an arrow  $\phi: x \rightarrow y$  over a morphism of schemes  $f: U \rightarrow V$ , the natural pullbacks:

$$\phi^*: \text{Aut}_{V, \mathcal{G}}(y) \rightarrow \text{Aut}_{U, \mathcal{G}}(x) \quad \text{and} \quad f^*: \mathcal{G}(V) \rightarrow \mathcal{G}(U)$$

commute with the isomorphisms, i.e.  $\alpha_x \circ f^* = \phi^* \circ \alpha_y$ .

**Definition 1.26.** A gerbe  $\mathcal{G}$  on  $\mathcal{X}$  is *trivial*, if the fibre category  $\mathcal{G}(\mathcal{X})$  is non-empty.

**Example 1.27.** The stack  $\mathcal{B}_{\mathcal{X}}G$  of  $G$ -principal bundles on  $\mathcal{X}$  is a  $G$ -gerbe and it is *the* trivial gerbe, i.e. every trivial  $G$ -gerbe on  $\mathcal{X}$  is isomorphic to it.

**Remark 1.28.** By definition each gerbe is locally non-empty, and therefore locally trivial.

**Example 1.29.** Let  $\mathcal{X}$  be a stack and  $\mathcal{X} \rightrightarrows H$  a  $H$ -rigidification of it, where  $H$  is a flat subgroup stack of the inertia of  $\mathcal{X}$ . Then  $\mathcal{X} \rightarrow \mathcal{X} \rightrightarrows H$  is a  $H$ -gerbe.

Let  $\mathcal{X}'$  be the  $r$ -th root of a line bundle over  $\mathcal{X}$ . The stack  $\mathcal{X}'$  is a gerbe banded by the constant sheaf  $\mu_r$ , especially  $\mathcal{X} = \mathcal{X}' \rightrightarrows \mu_r$ . Therefore  $\mathcal{X}'$  and  $\mathcal{X}$  have the same coarse moduli space.

An important result about  $G$ -gerbes is their classification up to isomorphisms that preserve the  $G$ -banded structure. By [14, Section IV.3.4], the group  $H_{\text{ét}}^2(\mathcal{X}, G)$  classifies the equivalence classes of  $G$ -gerbes on  $\mathcal{X}$ .

Let

$$0 \rightarrow \mu_r \xrightarrow{\iota} \mathbb{G}_m \xrightarrow{\wedge^r} \mathbb{G}_m \rightarrow 0$$

be the Kummer sequence, it induces a long exact sequence

$$\dots \rightarrow H_{\text{ét}}^1(\mathcal{X}, \mathbb{G}_m) \rightarrow H_{\text{ét}}^2(\mathcal{X}, \mu_r) \xrightarrow{\iota^*} H_{\text{ét}}^2(\mathcal{X}, \mathbb{G}_m) \rightarrow \dots$$

The  $\mu_r$ -gerbes whose  $H_{\text{ét}}^2(\mathcal{X}, \mu_r)$ -class has zero image via  $\iota^*$  in  $H_{\text{ét}}^2(\mathcal{X}, \mathbb{G}_m)$  are called *essentially trivial* and, from the exact sequence, they are exactly the ones obtained as  $r$ -th roots of a line bundle on  $\mathcal{X}$ .

Let  $\mathcal{G}$  be a  $\mu_r$ -gerbe on  $\mathcal{X}$ . The Leray spectral sequence for the étale sheaf  $\mathbb{G}_m$  with respect to the map  $\nu: \mathcal{G} \rightarrow \mathcal{X}$  gives the exact sequence:

$$0 \rightarrow H_{\text{ét}}^1(\mathcal{X}, \mathbb{G}_m) \xrightarrow{\nu^*} H_{\text{ét}}^1(\mathcal{G}, \mathbb{G}_m) \xrightarrow{\text{res}} H_{\text{ét}}^0(\mathcal{X}, R^1\nu_*\mathbb{G}_m) \xrightarrow{\text{obs}} H_{\text{ét}}^2(\mathcal{X}, \mathbb{G}_m) \rightarrow \dots$$

Here the map  $\text{res}: \text{Pic}(\mathcal{G}) \rightarrow \text{Pic}(\mathcal{B}_{\mu_r}) = \mathbb{Z}/r\mathbb{Z}$  is the restriction of line bundles to the fibre of  $\nu$ , that is isomorphic to  $\mathcal{B}_{\mu_r}$ . The map  $\text{obs}: \mathbb{Z}/r\mathbb{Z} \rightarrow H_{\text{ét}}^2(\mathcal{X}, \mathbb{G}_m)$  sends  $\bar{1}$  to the image by  $\iota^*$  of the class of the  $\mu_r$ -gerbe  $\mathcal{G}$ , thus it is zero if and only if the gerbe is essentially trivial and in this case the exact sequence becomes a short one.

## 2. CONSTRUCTION OF MD-STACKS

In the following, we propose the definition of an MD-stack, similar to the one of MD-spaces. We want to stress the slight differences. First, since MD-spaces have at most quotient singularities, we can associate to them Deligne-Mumford stacks which will be smooth. Moreover for a stack  $\mathcal{X}$ , we change the definition of its Cox ring to  $\mathcal{R}(\mathcal{X}) = \bigoplus_{\mathcal{L} \in \text{Pic}(\mathcal{X})} H^0(\mathcal{X}, \mathcal{L})$ . The structure as an algebra is obtained in the similar way as in Section 1.1:

If  $\text{Pic}(\mathcal{X})$  is finitely generated as a  $\mathbb{Z}$ -module, we can fix an isomorphism  $\text{Pic}(\mathcal{X}) \cong \mathbb{Z}/r_1\mathbb{Z} \times \dots \times \mathbb{Z}/r_k\mathbb{Z}$  with  $r_i \geq 0$ . To avoid confusion, we note that  $r_i$  can be zero, in which case  $\mathbb{Z}/r_i\mathbb{Z} = \mathbb{Z}$ . For each  $1 \leq i \leq k$ , choose a line bundle  $\mathcal{L}_i$  that generates the corresponding  $\mathbb{Z}/r_i\mathbb{Z}$ , and moreover an isomorphism  $\sigma_i: \mathcal{L}_i^{r_i} \xrightarrow{\sim} \mathcal{O}_{\mathcal{X}}$ .

Note that any line bundle  $\mathcal{L} \in \text{Pic}(\mathcal{X})$  can be written uniquely up to isomorphism as  $\bigotimes \mathcal{L}_i^{a_i}$  with  $0 \leq a_i < r_i$  for  $r_i \neq 0$ . Moreover, for  $r_i \neq 0$  the chosen isomorphisms  $\sigma_i$  induce isomorphisms  $\mathcal{L}_i^{a_i} \otimes \mathcal{L}_i^{b_i} \xrightarrow{\sim} \mathcal{L}_i^{c_i}$ , where

$0 \leq a_i, b_i, c_i < r_i$  and  $a_i + b_i = c_i \in \mathbb{Z}/r_i\mathbb{Z}$ . These data define the sheaf of algebras  $\overline{\mathcal{R}} = \bigoplus_{\mathcal{L} \in \text{Pic}(\mathcal{X})} \mathcal{L}$  on  $\mathcal{X}$ .

**Definition 2.1.** Let  $\mathcal{X}$  be a stack, such that  $\text{Pic}(\mathcal{X})$  is finitely generated. The *Cox ring* of  $\mathcal{X}$  is the  $\text{Pic}(\mathcal{X})$ -graded ring

$$\mathcal{R}(X) = H^0(X, \overline{\mathcal{R}}).$$

**Remark 2.2.** By [15, Proposition 4.7.3] and [16, Proposition 4.1.], the sheaf of  $\mathcal{O}_{\mathcal{X}}$ -algebras  $\overline{\mathcal{R}}$  defines a  $\text{Hom}(\text{Pic}(\mathcal{X}), \mathbf{k}^*)$ -torsor over  $\mathcal{X}$ .

Actually, there are many ways to define the algebra structure on  $\mathcal{R}(\mathcal{X})$  by choosing different (not necessarily minimal) generating sets of  $\text{Pic}(\mathcal{X})$ . All of them yield isomorphic Cox rings.

**Definition 2.3.** We call a Deligne-Mumford stack  $\mathcal{X}$  a *MD-stack* if

- (1)  $\mathcal{X}$  is smooth,
- (2)  $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^*) = \mathbf{k}^*$ ,
- (3)  $\text{Pic}(\mathcal{X})$  is finitely generated as a  $\mathbb{Z}$ -module,
- (4)  $\mathcal{R}(\mathcal{X})$  is finitely generated as a  $\mathbf{k}$ -algebra,

**Definition 2.4.** We call an MD-stack  $\mathcal{X}$  an *MD-quotient stack*, or just *MD-quotient* if  $\mathcal{X} = [\text{Spec } \mathcal{R}(\mathcal{X}) \setminus V(J_{\text{irr}}) / \text{Hom}(\text{Pic}(\mathcal{X}), \mathbf{k}^*)]$ .

**Remark 2.5.** For an orbifold, the Picard group and the divisor class group coincide. In the general case, the divisor class group can be strictly contained in the Picard group. As an example, for a positive integer  $r$ , the stack  $\mathcal{B}_{\mu_r}$  is an MD-quotient stack. Here  $\text{Pic}(\mathcal{B}_{\mu_r}) = \mathbb{Z}/r\mathbb{Z}$  which gives the quotient description, but  $\text{Cl}(\mathcal{B}_{\mu_r}) = 0$ .

**Remark 2.6.** For a reductive group scheme  $G$  acting linearly on some scheme  $Z$ , we can look at the map  $[Z^{ss}/G] \rightarrow Z^{ss}/G$ . If  $[Z^{ss}/G]$  is a Deligne-Mumford stack, then the scheme  $Z^{ss}/G$  is already the coarse moduli space, see for example [4].

Therefore an MD-space  $X = Z^{ss}/G$  with  $Z = \text{Spec } \mathcal{R}(X)$  and  $G = \text{Hom}(\text{Cl}(X), \mathbf{k}^*)$  gives a smooth Deligne-Mumford stack  $\mathcal{X} = [Z^{ss}/G]$ , if  $Z^{ss}$  is smooth. Moreover in this case,  $\mathcal{X}$  is the canonical stack of  $X$ , as the singular locus of  $X$  has codimension at least 2, and  $\mathcal{X} \rightarrow X$  is an isomorphism outside the singular locus.

Moreover it is a classical fact that every variety with finite quotient singularities is the coarse moduli space of a canonical smooth Deligne-Mumford stack unique up to isomorphism; see [12, Remark 4.9].

Our aim for the rest of the section is to study how the definitions of MD-stacks and MD-quotients behave with respect to the constructions we recalled in the previous section. First we describe the canonical stack of an MD-space and then we prove that root constructions and gerbes preserve the property of being an MD-stack.

**Theorem 2.7.** *Let  $X$  be an MD-space. Then its canonical stack  $\mathcal{X}^{can}$  is an MD-stack and  $\mathcal{R}(X) = \mathcal{R}(\mathcal{X}^{can})$ . Moreover, suppose that  $\text{Spec } \mathcal{R}(X) \setminus V(J_{\text{irr}})$  is smooth. Then its canonical stack  $\mathcal{X}^{can}$  is an MD-quotient, so*

$$\mathcal{X}^{can} = [\text{Spec } \mathcal{R}(\mathcal{X}^{can}) \setminus V(J_{\text{irr}}) / \text{Hom}(\text{Pic}(\mathcal{X}^{can}), \mathbf{k}^*)].$$

*Proof.* First, the canonical stack  $\mathcal{X}^{can}$ , that is smooth by definition, exists by Remark 2.6. By construction, we get  $\text{Pic}(\mathcal{X}^{can}) = \text{Cl}(\mathcal{X}^{can}) = \text{Cl}(X)$  which is finitely generated. Moreover, we note that  $H^0(X, \mathcal{O}(D)) = H^0(\mathcal{X}^{can}, \mathcal{O}(D))$  for any divisor class  $D$ , by [12, Section 1.2].

Therefore,  $H^0(\mathcal{X}^{can}, \mathcal{O}_{\mathcal{X}^{can}}^*) = \mathbf{k}^*$  and  $\mathcal{R}(\mathcal{X}^{can}) = \mathcal{R}(X)$  as algebras. For the latter note that, if we fix generators  $\mathcal{L}_i$  of  $\text{Pic}(\mathcal{X}^{can})$  as in the construction of the Cox ring  $\mathcal{R}(\mathcal{X}^{can})$ , then we can write an analogous resolution  $K \twoheadrightarrow \text{Cl}(X)$  by using the isomorphism  $\text{Pic}(\mathcal{X}^{can}) = \text{Cl}(X)$ , and vice versa. Hence  $\mathcal{X}^{can}$  is indeed an MD-stack.

By Lemma 1.12 it is immediate that  $J_{\text{irr}}(\mathcal{X}^{can}) = J_{\text{irr}}(X)$ . If we assume additionally that  $\text{Spec } \mathcal{R}(X) \setminus V(J_{\text{irr}}) = \text{Spec } \mathcal{R}(\mathcal{X}^{can}) \setminus V(J_{\text{irr}})$  is smooth, then  $\mathcal{X}^{can}$  is even an MD-quotient by Remark 2.6.  $\square$

In the following example we see that the assumption on the smoothness of  $\text{Spec } \mathcal{R}(X) \setminus V(J_{\text{irr}}(X))$  is indeed necessary to have the description of the canonical stack as an MD-quotient.

**Example 2.8.** Let  $\text{Dic}_n$  be the *dicyclic* or *binary dihedral* group of  $4n$  elements, defined as:

$$\text{Dic}_n = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \right\rangle = \langle s, r \mid s^4 = 1 = r^{2n}, r^n = s^2, srs^{-1} = r^{-2} \rangle$$

for a primitive  $2n$ -th root  $\xi$  of unity and let  $X = \mathbb{A}^2 / \text{Dic}_n$  be the quotient by the usual matrix action. It turns out that  $X$  is an MD-space. First, it has an isolated quotient singularity at zero. Easy calculations show that

$$\text{Cl}(X) = \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } n \text{ even,} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n \text{ is odd,} \end{cases}$$

$\mathcal{R}(X) = \mathbf{k}[u, v, w]/(u^n - vw)$  and that invertible global sections are just the non-zero constants. As  $V(J_{\text{irr}})$  is empty in this case, we computed that  $X = \text{Spec } \mathcal{R}(X) / \text{Hom}(\text{Cl}(X), \mathbf{k}^*) = \text{Spec } \mathcal{R}(X)^{\text{Cl}(X)}$ .

On the other hand, since  $\mathcal{R}(X)$  is singular in zero, the quotient stack  $\mathcal{X} = [\text{Spec } \mathcal{R}(X) / \text{Hom}(\text{Cl}(X), \mathbf{k}^*)]$  is not smooth. Hence  $\mathcal{X}$  differs from the (smooth) canonical stack  $\mathcal{X}^{can} = [\mathbb{A}^2 / \text{Dic}_n]$ .

The computation of the proof above shows that  $\mathcal{X}^{can}$  is an MD-stack with  $\text{Pic}(\mathcal{X}^{can}) = \text{Cl}(X)$  and  $\mathcal{R}(\mathcal{X}^{can}) = \mathcal{R}(X)$ . Consequently, there is no description of  $\mathcal{X}^{can}$  as an MD-quotient, since the candidate would be the singular  $\mathcal{X}$ .

**Theorem 2.9.** *Let  $\mathcal{X}$  be an MD-stack and  $\mathcal{X}'$  be a stack obtained by root constructions along simple normal crossing divisors from  $\mathcal{X}$ . Then  $\mathcal{X}'$  is an MD-stack and its Cox ring is of the form*

$$\mathcal{R}(\mathcal{X}') = \mathcal{R}(\mathcal{X})[z_1, \dots, z_l] / (z_i^{r_i} - s_i \mid i = 1, \dots, l)$$

for some positive integers  $r_i$  and  $s_i \in \mathcal{R}(\mathcal{X})$ .

*Proof.* Smoothness is ensured by Corollary 1.24.

Without loss of generality, we restrict to the case of a single smooth divisor  $\mathcal{D}$  on  $\mathcal{X}$  and show the remaining properties of MD-stacks for the stack  $\nu: \mathcal{X}' = \sqrt[l]{\mathcal{D}/\mathcal{X}} \rightarrow \mathcal{X}$ .

By [10, Theorem 3.1.1] we have the following pushout-diagram for the Picard groups:

$$\begin{array}{ccccc} 1 & & \mathbb{Z} & \xrightarrow{\cdot r} & \mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}(\mathcal{D}) & \text{Pic}(\mathcal{X}) & \xrightarrow{\nu^*} & \text{Pic}(\mathcal{X}') & \cong \text{Pic}(\mathcal{X}) \oplus \mathbb{Z} / \mathbb{Z} \cdot (\mathcal{O}(\mathcal{D}), -r) \end{array}$$

The line bundle  $\mathcal{T}$  on  $\mathcal{X}'$  corresponding to  $(\mathcal{O}, 1)$  is called the tautological line bundle. Moreover, there is an isomorphism  $\sigma: \mathcal{T}^r \xrightarrow{\sim} \nu^* \mathcal{O}(\mathcal{D})$  and a tautological section  $\tau$  of  $\mathcal{T}$  such that  $\sigma(\tau^r) = \nu^*(s)$ , where  $s$  is the section cutting out  $\mathcal{D}$ .

Consequently, for any line bundle  $\mathcal{L}'$  on  $\mathcal{X}'$ , there exists a line bundle  $\mathcal{L}$  on  $\mathcal{X}$ , unique up to isomorphism, and a unique integer  $0 \leq k < r$ , such that  $\mathcal{L}' \cong \nu^* \mathcal{L} \otimes \mathcal{T}^k$ . Moreover by [10, Corollary 3.1.3], the multiplication by  $\tau^k$  gives an isomorphism between global sections:

$$H^0(\mathcal{X}, \mathcal{L}) \xrightarrow{\sim} H^0(\mathcal{X}', \nu^* \mathcal{L}) \xrightarrow[\cdot \tau^k]{\sim} H^0(\mathcal{X}', \mathcal{L}').$$

It is obvious that  $\nu_* \mathcal{O}_{\mathcal{X}'} \cong \mathcal{O}_{\mathcal{X}}$ , therefore  $H^0(\mathcal{X}', \mathcal{O}_{\mathcal{X}}^*) = H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^*) = \mathbf{k}^*$ .

Next we show that the Cox ring  $\mathcal{R}(\mathcal{X}')$  is finitely generated. Choose line bundles  $\mathcal{L}_1, \dots, \mathcal{L}_k$  on  $\mathcal{X}$  and isomorphisms  $\sigma_i: \mathcal{L}_i^{s_i} \xrightarrow{\sim} \mathcal{O}_{\mathcal{X}}$  to define the algebra structure on the Cox ring of  $\mathcal{X}$ . Since  $\text{Pic}(\mathcal{X}')$  is generated by  $\nu^* \mathcal{L}$  and  $\mathcal{T}$ , we can use  $\nu^* \sigma_i$  and  $\sigma: \mathcal{T}^r \xrightarrow{\sim} \nu^* \mathcal{O}(\mathcal{D})$  to obtain the algebra structure on  $\mathcal{R}(\mathcal{X}')$ . With these data the Cox Ring of  $\mathcal{X}'$  is  $\mathcal{R}(\mathcal{X}') = \mathcal{R}(\mathcal{X})[z]/(z^r - s)$ .

Hence  $\mathcal{X}'$  is indeed an MD-stack.  $\square$

**Theorem 2.10.** *Let  $\mathcal{X}$  be a stack and  $\nu: \mathcal{X}' \rightarrow \mathcal{X}$  a  $H$ -gerbe on  $\mathcal{X}$ , where  $H$  is an abelian group scheme. Then  $\mathcal{X}$  is an MD-stack if and only if  $\mathcal{X}'$  is an MD-stack. Moreover in this case, their Cox rings are isomorphic.*

*Proof.* The local triviality of gerbes assures that  $\mathcal{X}$  is smooth if and only if  $\mathcal{X}'$  is smooth.

The Leray spectral sequence for  $\nu: \mathcal{X}' \rightarrow \mathcal{X}$  yields the long exact sequence:

$$0 \rightarrow \text{Pic}(\mathcal{X}) \xrightarrow{\nu^*} \text{Pic}(\mathcal{X}') \rightarrow \text{Pic}(\mathcal{B}H) \rightarrow H^2(\mathcal{X}, \mathbb{G}_m)$$

and  $\text{Pic}(\mathcal{X}')$  turns out to be an extension of  $\text{Pic}(\mathcal{X})$  with a subgroup of the finitely generated group  $\text{Pic}(\mathcal{B}H)$ . Therefore, if one of the Picard groups of  $\mathcal{X}$  and  $\mathcal{X}'$  is finitely generated as a  $\mathbb{Z}$ -module, the other is, too.

Locally, let  $\mathcal{X} = \text{Spec } A$ . Since  $\mathcal{X}'$  is a  $H$ -gerbe on  $\mathcal{X}$ , it is locally trivial, and, maybe after shrinking the affine set, we can assume that  $\mathcal{X}' \times_{\mathcal{X}} \text{Spec } A \cong \mathcal{B}H \times \text{Spec } A$ . Locally, the exact sequence above becomes

$$0 \rightarrow \text{Pic}(\mathcal{X}) \xrightarrow{\nu^*} \text{Pic}(\mathcal{X}') \rightarrow \text{Pic}(\mathcal{B}H) \rightarrow 0.$$

The line bundles on  $\mathcal{X}'$  of the form  $\mathcal{L}' = \nu^* \mathcal{L}$ , with  $\mathcal{L} \in \text{Pic}(\mathcal{X})$ , correspond to graded  $A$ -modules concentrated in degree zero and for them we have  $H^0(\mathcal{X}, \mathcal{L}) = H^0(\mathcal{X}', \mathcal{L}')$ . Let now  $\mathcal{L}'$  be a line bundle on  $\mathcal{X}'$  not be mapped to zero in  $\text{Pic}(\mathcal{B}H)$ . Thus it corresponds to a graded  $A$ -module with trivial degree zero part and therefore  $H^0(\mathcal{X}', \mathcal{L}') = 0$ .

It is obvious that  $\nu_* \mathcal{O}_{\mathcal{X}'} \cong \mathcal{O}_{\mathcal{X}}$ , so  $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^*) = H^0(\mathcal{X}', \mathcal{O}_{\mathcal{X}'}^*)$ .

Moreover  $\mathcal{R}(\mathcal{X}') = \mathcal{R}(\mathcal{X})$  as a  $\mathbf{k}$ -algebra. Indeed, since only the line bundles pulled back from  $\mathcal{X}$  have non-trivial sections, to calculate the Cox Ring of  $\mathcal{X}'$  one can use the pullback of the line bundles and of the isomorphisms to define the Cox ring of  $\mathcal{X}$ .

So  $\mathcal{X}$  is an MD-stack if and only if  $\mathcal{X}'$  is.  $\square$

**Corollary 2.11.** *Let  $\mathcal{X}$  be a stack and  $\mathcal{X}'$  be a stack obtained by root constructions with line bundles on  $\mathcal{X}$ . Then  $\mathcal{X}$  is an MD-stack if and only if  $\mathcal{X}'$  is an MD-stack.*

**Theorem 2.12.** *Let  $\mathcal{X}$  be an MD-quotient stack. Then any stack  $\mathcal{X}'$  obtained by root constructions along simple normal crossing divisors and with line bundles from  $\mathcal{X}$  is again an MD-quotient.*

*Proof.* Let  $\nu: \mathcal{X}' \rightarrow \mathcal{X}$  be the induced map. By the Lemmata 1.21 and 1.12, the line bundle  $\nu^*\mathcal{L}$  is still ample and the ideal  $J'_{\text{irr}} = \sqrt{\langle H^0(\mathcal{X}', \nu^*\mathcal{L}) \rangle}$  in  $\mathcal{R}(\mathcal{X}')$  gives the irrelevant ideal of  $\mathcal{X}'$ . Putting together the results about the Cox rings and the Picard groups of the stacks obtained as roots from  $\mathcal{X}$  given in the proofs of Theorems 2.9 and 2.11, the desired description of  $\mathcal{X}'$  as a quotient follows now directly from Proposition 1.16.  $\square$

### 3. CHARACTERISATION OF MD-STACKS

In this section, we give a characterisation of MD-stacks among smooth Deligne-Mumford stacks and study the conditions under which an MD-stack can be obtained by roots from its canonical stack which turns out to be an MD-stack, too. The representation of MD-stacks as MD-quotients will be in our main focus.

We start our analysis of the relations between MD-stacks and their moduli spaces with the following result.

**Theorem 3.1.** *Let  $\mathcal{X}$  be an MD-stack and let  $X$  be its coarse moduli scheme. Then  $X$  is an MD-space.*

*Proof.* Let  $\mathcal{X}$  be an MD-stack and  $\pi: \mathcal{X} \rightarrow X$  the map to its coarse moduli space. Since  $\mathcal{X}$  is a smooth Deligne-Mumford stack, its coarse moduli space  $X$  is normal and has at most quotient singularities.

We prove that  $\text{Cl}(X)$  is finitely generated. Since  $\text{Cl}(X) = \text{Cl}(\mathcal{X}^{\text{can}}) = \text{Pic}(\mathcal{X}^{\text{can}})$ , we have just to compare  $\text{Pic}(\mathcal{X}^{\text{can}})$  and  $\text{Pic}(\mathcal{X})$ . The Leray spectral sequence applied to the map  $\pi_{\text{can}}: \mathcal{X} \rightarrow \mathcal{X}^{\text{can}}$  and to the sheaf  $\mathcal{O}_{\mathcal{X}}^*$  gives the exact sequence:

$$0 \rightarrow H^1(\mathcal{X}^{\text{can}}, \pi_{\text{can}*}\mathcal{O}_{\mathcal{X}}^*) \rightarrow H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^*) \rightarrow H^0(\mathcal{X}^{\text{can}}, R^1\pi_{\text{can}*}\mathcal{O}_{\mathcal{X}}^*) \rightarrow \dots$$

and then the injective morphism  $H^1(\mathcal{X}^{\text{can}}, \pi_{\text{can}*}\mathcal{O}_{\mathcal{X}}^*) = H^1(\mathcal{X}^{\text{can}}, \mathcal{O}_{\mathcal{X}^{\text{can}}}^*) = \text{Pic}(\mathcal{X}^{\text{can}}) \hookrightarrow H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^*) = \text{Pic}(\mathcal{X})$ . Since  $\mathcal{X}$  is an MD-stack,  $\text{Pic}(\mathcal{X})$  is finitely generated and the same is true for  $\text{Cl}(X)$ .

It is obvious that  $\pi_*\mathcal{O}_X \cong \mathcal{O}_{\mathcal{X}}$ , so  $H^0(X, \mathcal{O}_X^*) = H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^*) = \mathbf{k}^*$ .

Let us prove that the Cox ring of  $X$  is a finitely generated  $\mathbf{k}$ -algebra. Since  $H^0(\mathcal{X}^{\text{can}}, \mathcal{L}) = H^0(\mathcal{X}, \pi_{\text{can}}^*\mathcal{L})$  and since the tensor product commutes with pullback,  $\mathcal{R}(X) = \mathcal{R}(\mathcal{X}^{\text{can}})$  is a subalgebra of  $\mathcal{R}(\mathcal{X})$ .

Note that by [21, Lemma 3.2], for any homogeneous  $r \in \mathcal{R}(\mathcal{X})$  there is a  $d > 0$  such that  $r^d \in \mathcal{R}(X)$ . Actually there exists a  $d > 0$  which works for

all homogeneous elements in  $\mathcal{R}(\mathcal{X})$ . Choose a finite set of homogeneous generators  $r_1, \dots, r_m$  of  $\mathcal{R}(\mathcal{X})$ . Then  $B = \{\prod_i r_i^{j_i} \mid 1 \leq j_i < d\}$  generates  $\mathcal{R}(\mathcal{X})$  as a  $\mathcal{R}(X)$ -module. Indeed, write any element of  $\mathcal{R}(\mathcal{X})$  as a polynomial in the  $r_i$  with coefficients in  $\mathbf{k}$ . We can write any factor  $r_i^a$  of a monomial as  $r_i^{bd} \cdot r_i^c$  with  $0 \leq c < d$ . Note that  $r_i^{bd}$  is already in  $\mathcal{R}(X)$ . So the polynomial becomes a linear combination of elements in  $B$  with coefficients in  $\mathcal{R}(X)$ .

So we have a chain of inclusions  $\mathbf{k} \hookrightarrow \mathcal{R}(X) \hookrightarrow \mathcal{R}(\mathcal{X})$  with  $\mathcal{R}(\mathcal{X})$  finitely generated as a  $\mathcal{R}(X)$ -module. Hence we can apply [6, Proposition 7.8], so  $\mathcal{R}(X)$  is finitely generated as a  $\mathbf{k}$ -algebra.  $\square$

For the characterisation of MD-orbifolds the following general theorem plays a crucial role.

**Proposition 3.2** ([13, Theorem 6.1]). *Let  $\mathcal{X}$  be a smooth Deligne-Mumford orbifold, whose ramification divisor is simple normal crossing. Then  $\mathcal{X}$  can be obtained by roots from its canonical stack  $\mathcal{X}^{can}$ .*

To avoid confusions, we want to note that [13, Theorem 6.1] is actually formulated for arbitrary ramification divisors. In such a case the root constructions with these divisors may yield a singular stack, which is then approximated with a smooth stack. This additional step is not necessary here.

**Lemma 3.3** ([13, Lemma 5.4]). *Let  $U$  be a smooth algebraic space and  $G$  a diagonalisable group scheme (e.g. subgroup of a torus) which acts properly on  $U$  with finite stabilisers. Then the ramification divisor on  $[U/G]$  is simple normal crossing.*

Theorem 3.1 and Proposition 3.2 imply that smooth Deligne-Mumford orbifolds, with simple normal crossing ramification divisor, are MD-orbifolds exactly when their coarse moduli spaces are MD-spaces. In combination with Theorem 2.10 we have the following result, that characterise MD-stacks among the smooth Deligne-Mumford stacks.

**Corollary 3.4.** *Let  $\mathcal{X}$  be a smooth Deligne-Mumford stack with abelian generic stabiliser, whose ramification divisor is simple normal crossing. Then  $\mathcal{X}$  is an MD-stack if and only if its coarse moduli space  $X$  is an MD-space.*

Our last aim is to understand when an MD-stack can be obtained by roots from its canonical stack.

**Theorem 3.5.** *Let  $\mathcal{X} = [\mathrm{Spec} \mathcal{R}(\mathcal{X}) \setminus V(J_{\mathrm{irr}}) / \mathrm{Hom}(\mathrm{Pic}(\mathcal{X}), \mathbf{k}^*)]$  be an MD-quotient stack. Then it can be obtained by roots from its canonical stack  $\mathcal{X}^{can}$ .*

*Proof.* Denote with  $Z = \mathrm{Spec} \mathcal{R}(\mathcal{X}) \setminus V(J_{\mathrm{irr}})$  and  $G = \mathrm{Hom}(\mathrm{Pic}(\mathcal{X}), \mathbf{k}^*)$ . Since  $\mathcal{X}$  is smooth,  $Z$  is also smooth and by Lemma 3.3, the ramification divisor of  $\mathcal{X}$  is simple normal crossing.

Let  $\nu: \mathcal{X} \rightarrow \mathcal{X}^{rig}$  be the natural map from  $\mathcal{X}$  to its rigidification. Since the rigidification preserves the ramification divisor, by Proposition 3.2 the orbifold  $\mathcal{X}^{rig}$  is obtained by roots of divisors from its canonical stack  $\mathcal{X}^{can}$ .

Note that, by Theorem 2.10, we have also  $Z = \mathrm{Spec} \mathcal{R}(\mathcal{X}^{rig}) \setminus V(J_{\mathrm{irr}})$ . By Corollary 1.24, the smoothness of  $Z$  implies the smoothness of  $\mathrm{Spec} \mathcal{R}(\mathcal{X}^{can}) \setminus$

$V(J_{\text{irr}})$  and then of  $\text{Spec } \mathcal{R}(X) \setminus V(J_{\text{irr}})$ , where  $X$  is the coarse moduli space of these stacks.

By Theorems 2.7 and 2.12, the stack  $\mathcal{X}^{rig}$  is an MD-quotient of the form:

$$\mathcal{X}^{rig} = [Z / \text{Hom}(\text{Pic}(\mathcal{X}^{rig}), \mathbf{k}^*)],$$

we denote with  $G^{rig} = \text{Hom}(\text{Pic}(\mathcal{X}^{rig}), \mathbf{k}^*)$ .

Recall that the  $G$ -action on  $Z$  is given by the  $\text{Pic}(\mathcal{X})$ -grading on  $\mathcal{R}(\mathcal{X})$ . The proof of Theorem 2.10 shows that the grading of  $\mathcal{R}(\mathcal{X}) = \mathcal{R}(\mathcal{X}^{rig})$  are related through the pullback  $\nu^*: \text{Pic}(\mathcal{X}^{rig}) \rightarrow \text{Pic}(\mathcal{X})$ . So,  $\mathcal{R}(\mathcal{X}^{rig})$  is actually  $\text{Pic}(\mathcal{X})$ -graded, i.e.  $G$  acts on  $Z$  via the map  $G \rightarrow G^{rig}$  given by the dual of the pullback.

By construction the map  $\nu: \mathcal{X} \rightarrow \mathcal{X}^{rig}$  is a  $H$ -gerbe, where  $H$  is an abelian group scheme, and the Leray spectral sequence for  $\nu$  assures the existence of the following exact sequence

$$0 \rightarrow \text{Pic}(\mathcal{X}^{rig}) \xrightarrow{\nu^*} \text{Pic}(\mathcal{X}) \rightarrow \text{Pic}(\mathcal{B}H) \rightarrow H^2(\mathcal{X}^{rig}, \mathbb{G}_m).$$

Assuming that  $\text{coker } \nu^* = \bigoplus_{i=1}^n \mathbb{Z}/r_i\mathbb{Z}$ , we get the following short exact sequence

$$0 \rightarrow \text{Pic}(\mathcal{X}^{rig}) \rightarrow \text{Pic}(\mathcal{X}) \rightarrow \bigoplus_{i=1}^n \mathbb{Z}/r_i\mathbb{Z} \rightarrow 0$$

and, dualizing it, the short exact sequence:

$$0 \rightarrow \prod_{i=1}^n \mu_{r_i} \rightarrow G \rightarrow G^{rig} \rightarrow 0.$$

Choose now line bundles  $\mathcal{M}_i \in \text{Pic}(\mathcal{X})$  which are mapped to a primitive generator in  $\bar{1} \in \mathbb{Z}/r_i\mathbb{Z}$ , and to 0 otherwise. Then  $\mathcal{M}^{r_i} = \mathcal{L}_i$  for a line bundle  $\mathcal{L}_i \in \text{Pic}(\mathcal{X}^{rig})$ . Perform iteratively the roots of these line bundles and denote  $\mathcal{X}' = \sqrt[r]{\underline{\mathcal{L}}/\mathcal{X}^{rig}}$ , where  $\underline{r} = (r_1, \dots, r_n)$  and  $\underline{\mathcal{L}} = (\mathcal{L}_1, \dots, \mathcal{L}_n)$ . By Proposition 1.16,  $\mathcal{X}' = [Z/G']$ , where  $G'$  is defined by the following cartesian diagram

$$\begin{array}{ccc} G' & \longrightarrow & (\mathbf{k}^*)^n \\ \downarrow & & \downarrow \underline{r} \\ G^{rig} & \xrightarrow{\chi(\underline{\mathcal{L}})} & (\mathbf{k}^*)^n, \end{array} \quad (4)$$

in which  $\chi(\underline{\mathcal{L}})$  is given the characters of  $\mathcal{L}_i$ , and the action of  $G'$  on  $Z$  is defined via the map  $G' \rightarrow G^{rig}$ . By construction  $G$  fits in the following diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H' & \longrightarrow & G & \longrightarrow & G^{rig} \longrightarrow 0 \\ & & \parallel & & \downarrow \chi(\underline{\mathcal{M}}) & & \downarrow \chi(\underline{\mathcal{L}}) \\ 0 & \longrightarrow & H' & \longrightarrow & (\mathbf{k}^*)^n & \xrightarrow{\wedge_{\underline{r}}} & (\mathbf{k}^*)^n \longrightarrow 0, \end{array} \quad (5)$$

where  $H' = \prod_{i=1}^n \mu_{r_i}$  and again  $\chi$  indicates the characters of line bundles. This implies that the right square in the diagram is cartesian and that  $G \cong G'$ . Moreover the actions of  $G$  and  $G'$  on  $Z$  are both given through the map to  $G^{rig}$ , thus  $\mathcal{X} = \mathcal{X}'$  is obtained by roots of line bundles from  $\mathcal{X}^{rig}$ , that is itself obtained by roots of divisors from  $\mathcal{X}^{can}$ .  $\square$



**Remark 3.6.** We want to note that the previous theorem implies also that  $\mathcal{X}^{can}$  is an MD-quotient. As a consequence  $\mathcal{R}(\mathcal{X}^{can}) \setminus V(J_{irr}) = \mathcal{R}(X) \setminus V(J_{irr})$  is smooth, where  $X$  is the coarse moduli space.

#### 4. MD-STACKS AND SMOOTH TORIC DELIGNE-MUMFORD STACKS

The following propositions are generalisations of [18, Corollary 2.10] and [18, Proposition 2.11]. We want to remind the reader, that a *smooth toric Deligne-Mumford stack*  $\mathcal{X}$  is a smooth separated Deligne-Mumford stack with an action of a Deligne-Mumford torus  $\mathcal{T}$ , which is contained as an open dense orbit; see [12]. Here a Deligne-Mumford torus  $\mathcal{T}$  means that  $\mathcal{T}$  is isomorphic to  $T \times \mathcal{B}G$ , where  $T$  is an ordinary torus and  $G$  a finite abelian group.

**Proposition 4.1.** *Let  $\mathcal{X}$  be an MD-quotient stack. Then  $\mathcal{X}$  is a smooth toric Deligne-Mumford stack if and only if its Cox ring is a polynomial ring.*

*Proof.* First we assume that  $\mathcal{R}(\mathcal{X}) = \mathbf{k}[x_1, \dots, x_n]$ . Therefore  $T = (\mathbf{k}^*)^n$  acts naturally on  $\text{Spec } \mathcal{R}(\mathcal{X}) = \mathbb{A}^n$  by componentwise multiplication. Fix the presentation  $\mathbb{Z}^n \twoheadrightarrow \text{Pic}(\mathcal{X})$  where  $e_i$  is sent to the divisor class  $[D_i]$  given by  $\{x_i = 0\}$ . By applying  $\text{Hom}(-, \mathbf{k}^*)$  to this surjection, we get an inclusion of  $G = \text{Hom}(\text{Pic}(\mathcal{X}), \mathbf{k}^*)$  into  $T$ . Note that the action of  $G$  is given as a subgroup of  $T$ .

Furthermore, the locus  $V(J_{irr})$  is cut out by some of the hyperplanes  $\{x_i = 0\}$ , since we can choose the ample line bundle to be given as  $\mathcal{O}(D)$  with  $D = \sum a_i D_i$ , so the global sections of  $\mathcal{O}(D)$  have a basis consisting of monomials in  $x_i$ . Therefore,  $V(J_{irr})$  is  $T$ -invariant and consequently  $T$  acts on  $Z = \text{Spec } \mathcal{R}(\mathcal{X}) \setminus V(J_{irr})$ .

This action descends to an action of  $[T/G]$  on  $[Z/G]$ .  $[T/G]$  has again a torus  $T'$  as its coarse moduli space (of dimension  $n - \text{rk } G$ ), so by [12, Section 6.1]  $[T/G]$  is a Deligne-Mumford torus, i.e.  $[T/G] = T' \times \mathcal{B}H$  for some finite abelian group  $H$ . Moreover, since  $T$  is open and dense in  $Z$ , the same holds true for  $[T/G]$  inside  $[Z/G]$ . Hence  $[Z/G]$  is a smooth toric Deligne-Mumford stack.

The converse direction was shown in [12, Theorem 7.7].  $\square$

**Proposition 4.2.** *Let  $\mathcal{X}$  be an MD-quotient stack such that divisor class group of its coarse moduli space is free. Then there is a closed embedding  $\iota: \mathcal{X} \hookrightarrow \mathcal{Y}$  with  $\mathcal{Y}$  a smooth toric Deligne-Mumford stack such that  $\iota^*: \text{Pic}(\mathcal{Y}) \xrightarrow{\sim} \text{Pic}(\mathcal{X})$ .*

*Proof.* First we note that  $\mathcal{X}$  can be obtained by roots from its canonical stack  $\mathcal{X}^{can}$ , by Theorem 3.5.

Let  $X$  be the coarse moduli space of  $\mathcal{X}$ . For the Cox ring  $\mathcal{R}(X)$  choose a presentation  $\mathcal{R}(X) = \mathbf{k}[x_1, \dots, x_m]/I$  where  $I$  is a  $\text{Cl}(X)$ -homogeneous ideal and where  $\{x_i = 0\}$  define divisors  $D_i$  on  $X$ . Moreover, we assume that among these divisors  $D_i$  there are those with which the root constructions are performed to obtain  $\mathcal{X}$  (after pulling them back to  $\mathcal{X}^{can}$ ).

Since  $\text{Cl}(X)$  is assumed to be free, by [7, Proposition 5.2], there is a toric variety  $Y$  and a closed embedding  $X \hookrightarrow Y$  inducing an isomorphism  $\text{Cl}(Y) \xrightarrow{\sim} \text{Cl}(X)$  by pullback. The inclusion  $X \hookrightarrow Y$  is induced by the

presentation  $\mathcal{R}(Y) = \mathbf{k}[x_1, \dots, x_m] \twoheadrightarrow \mathcal{R}(X)$ . The inclusion  $X \hookrightarrow Y$  gives also an inclusion  $\mathcal{X}^{can} \hookrightarrow \mathcal{Y}^{can}$  with  $\mathrm{Pic}(\mathcal{Y}^{can}) \xrightarrow{\sim} \mathrm{Pic}(\mathcal{X}^{can})$ , by Remark 3.6 and Theorem 2.7.

Now we perform the root constructions on  $\mathcal{X}^{can}$  along the divisors to obtain  $\mathcal{X}^{rig}$ . By performing the analogous root constructions on  $\mathcal{Y}^{can}$ , we get a smooth toric Deligne-Mumford orbifold which we denote by  $\mathcal{Y}^{rig}$ . It is clear that the inclusion lifts to  $\mathcal{X}^{rig} \hookrightarrow \mathcal{Y}^{rig}$  and  $\mathrm{Pic}(\mathcal{Y}^{rig}) \xrightarrow{\sim} \mathrm{Pic}(\mathcal{X}^{rig})$ .

Finally, we perform the root constructions with line bundles to obtain  $\mathcal{X}$  from  $\mathcal{X}^{rig}$ . If we do the same constructions starting with  $\mathcal{Y}^{rig}$ , we arrive at the statement.  $\square$

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